

## STRONG FATOU-1-POINTS OF BLASCHKE PRODUCTS

BY

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**ABSTRACT.** This paper shows that to every countable set  $M$  on the unit circle there corresponds a Blaschke product whose set of strong Fatou-1-points contains  $M$ . It also shows that some Blaschke products have an uncountable set of strong Fatou-1-points.

**1. Introduction.** Let  $f$ ,  $\zeta$  and  $\omega$  denote a complex-valued function in the unit disk  $D$ , a point on the unit circle  $C$ , and a complex number, possibly the point at infinity. We call  $\zeta$  a *Fatou- $\omega$ -point* of  $f$  if in each Stolz angle at  $\zeta$  the value of  $f(z)$  tends to  $\omega$  as  $z \rightarrow \zeta$ . We call  $\zeta$  a *strong Fatou- $\omega$ -point* if, in addition,  $|\omega| = 1$  and  $f$  maps each Stolz angle at  $\zeta$  into a Stolz angle at  $\omega$ .

The motivation for our investigation arose in a study of the Fatou points of strongly annular functions, that is, of holomorphic functions in  $D$  whose minimum modulus on a sequence of concentric circles  $C_n$  tends to  $\infty$  as  $n \rightarrow \infty$ . An example of a strongly annular function  $g$  having a Fatou- $\infty$ -point was known. If a Blaschke product  $B$  has a strong Fatou-1-point at each point of a set  $E$  on the circle  $C$ , and if  $B$  maps certain circles with center 0 into sufficiently small neighborhoods of the circles  $C_n$ , then the composite function  $g \circ B$  is strongly annular and has a Fatou limit  $\infty$  at each point of  $E$ .

For the construction of annular functions with many Fatou points, we found a method more appropriate than composition of functions [1]; but the problem of strong Fatou-1-points of a Blaschke product appears to be worthy of analysis in its own right.

A holomorphic function  $f$  in  $D$  with a Fatou-1-point  $\zeta$  on  $C$  has a *finite angular derivative* at  $\zeta$  if the difference quotient  $[f(z) - 1]/[z - \zeta]$  approaches a finite limit as  $z \rightarrow \zeta$  in Stolz angles. Evidently, if  $\chi$  is a chord of  $D$  with endpoints  $\eta$  and  $\zeta$ , then the existence of a finite angular derivative of  $f$  at  $\zeta$  implies

$$(1.1) \quad \arg[f(z) - 1] \rightarrow \arg[\eta/\zeta - 1] \quad \text{as } z \rightarrow \zeta \text{ along } \chi$$

and the convergence is uniform as long as  $\eta$  is bounded away from  $\zeta$ . On the other hand, even if  $f$  is a Blaschke product, it need not have a finite angular derivative at each Fatou-1-point. To see this let  $a_1, a_2, \dots$  be an infinite Blaschke sequence lying on the oricycle  $(1 - |z|^2)/|1 - z|^2 = 1$ , and let  $B$  denote the Blaschke product with

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Received by the editors November 15, 1982.

1980 *Mathematics Subject Classification*. Primary 30D50; Secondary 30D40.

<sup>1</sup> Belna and Piranian gratefully acknowledge support from the National Science Foundation.

zeros  $a_1, \bar{a}_1, a_2, \bar{a}_2, \dots$ . Because the sum of the terms  $(1 - |a_n|^2)/|1 - a_n|^2$  is not finite, it follows from a result of O. Frostman [4, p. 177] that  $B$  does not have a finite angular derivative at 1. Because  $B(z)$  is real on the interval  $(0, 1)$ ,  $B$  has the radial limit 1 at 1 if  $a_n \rightarrow 1$  rapidly. Then, since  $B$  shrinks hyperbolic distances, 1 is a strong Fatou-1-point of  $B$ .

Corresponding to each holomorphic function  $f$  in  $D$ , we denote by  $S(f)$  the set of strong Fatou-1-points of  $f$  and by  $S^*(f)$  the subset of  $S(f)$  where, in addition,  $f$  has a finite angular derivative. §2 of this paper deals with the size of  $S^*(f)$  in the case where  $|f(z)| < 1$  throughout  $D$ . We introduce an extended notion of the finite angular derivative, and we show that to each countable set  $\{\zeta_n\}$  on  $C$  there corresponds a Blaschke product  $B$  having at each point of  $\{\zeta_n\}$  a strong Fatou-1-point and a finite angular derivative in the extended sense. If  $\{\zeta_n\}$  is of type  $G_\delta$  we can construct the function  $B$  so that  $\{\zeta_n\} = S^*(B)$ . Considering the function  $g(z) = \arg[B(z) - 1]$  and using (1.1), we may apply a result of H. Blumberg [2, p. 18] to deduce that the set  $S^*(B)$  is countable for every Blaschke product  $B$ . By other methods M. Heins proved a theorem (see [5, Theorem 8.1]) from which it follows that if  $f$  is a holomorphic mapping of  $D$  into  $D$ , then  $f$  has at most countably many Fatou-1-points where the angular derivative exists. For the sake of completeness we insert a sketch of a slightly different proof of this. At each point  $\zeta$  of  $S^*(f)$  we construct a triod consisting of the line segment with endpoints  $\zeta$  and  $2\zeta$  and of two segments that terminate at  $\zeta$  and make an angle  $3\pi/4$  with the first segment. Since  $f$  has a finite angular derivative at each point  $\zeta$  of  $S^*(f)$ , the relation  $f(D) \subset D$  requires that for each point  $\zeta$  in  $S^*(f)$  there exists a positive number  $k_\zeta$  such that

$$\frac{f(z) - 1}{z/\zeta - 1} \rightarrow k_\zeta$$

as  $z \rightarrow \zeta$  in a Stolz angle. This implies that near the point  $\zeta$  the imaginary part of  $f(z)$  is positive (negative) on the segment lying to the left (right) of the radius vector of  $\zeta$  as seen from the origin. Elementary computations show that if none of our segments in  $D$  has length greater than  $1/2$ , then no two left segments and no two right segments intersect in  $D$ . If, in addition, we shorten each of the segments in  $D$  sufficiently so that the imaginary part of  $f(z)$  has constant sign on the segment, then no left segment intersects any of the right segments, and therefore the triods of our system are disjoint. By a theorem of R. L. Moore [6], the set  $S^*(f)$  is countable.

In §3 we show that if  $\{\zeta_n\}$  is a countable set on  $C$  and  $\{\omega_n\}$  is a sequence of points on  $C$ , then some Blaschke product  $B$  has for each index  $n$  a strong Fatou- $\omega_n$ -point at  $\zeta_n$ . This result gives a partial solution to a problem mentioned by G. T. Cargo [3, p. 288]. Our construction involves successive adjustments of the positions of the zeros of  $B$ . We hope that our treatment of the technical difficulties will be applicable in other contexts.

In §4 we show that there exists a Blaschke product  $B$  with uncountably many strong Fatou-1-points. Explicitly,  $B = (G + 1)/(G - 1)$ , where

$$G(z) = \int_{-\pi}^{\pi} \frac{z + e^{it}}{z - e^{it}} d\mu(t)$$

and  $d\mu$  is the standard singular distribution of unit mass on the image  $M$  of the classical Cantor set under the obvious mapping of the interval  $[0, 1]$  onto the arc of  $C$  whose length is 1 and whose midpoint is 1. The set  $M$  is precisely the set  $S(B)$ . The proof makes no appeal to the existence of finite angular derivatives at the points of  $M$  (obviously), but rather to the fact that the images under  $G$  of the radii terminating at the points of  $M$  all lie in a certain wedge in the left half-plane.

**2. Equal Fatou limits.** By a *boundary domain* at 1 we shall mean a convex domain  $H$  whose boundary lies in  $D$  except for the point 1. If  $\zeta \in C$  and  $H$  is a boundary domain, we denote by  $\zeta H$  the image of  $H$  under the rotation  $z \rightarrow \zeta z$ . A boundary domain is *tangent* if its boundary is tangent to  $C$ . We shall say that the *restricted derivative* of  $f$  relative to a boundary domain  $\zeta H$  (briefly: the  $H$ -derivative of  $f$ ) exists if for some number  $w$  the quotient  $[f(z) - w]/[z - \zeta]$  approaches a finite limit as  $z \rightarrow \zeta$  in  $\zeta H$ . Our discussion in §1 shows that if  $H$  is a tangent boundary domain, then the set of Fatou-1-points where an inner function has an  $H$ -derivative is at most countable.

**THEOREM 1.** *Corresponding to each countable set  $\{\zeta_n\}$  on  $C$  and each tangent boundary domain  $H$  at 1, there exists a Blaschke product whose Fatou limit is 1 and whose  $H$ -derivative exists at each point  $\zeta_n$ . If the set  $\{\zeta_n\}$  is of type  $G_8$ , there exists a Blaschke product  $B$  such that  $\{\zeta_n\} = S(B)$  and the  $H$ -derivative of  $B$  exists at each point  $\zeta_n$ .*

In the first part of the proof we merely suppose that  $\{\zeta_n\}$  is a countable set on  $C$ .

Let  $G(z) = \sum a_n(z + \zeta_n)/(z - \zeta_n)$ , where  $\{a_n\}$  is a positive sequence such that  $\sum a_n < \infty$ . Then  $G$  maps  $D$  into the left half-plane  $L$ ; because the function  $F(w) = (w + 1)/(w - 1)$  maps  $L$  into  $D$ , the composite function  $B = F \circ G$  maps  $D$  into  $D$ .

In the domain  $\zeta_n H$  each term of index less than  $m$  in the series for  $G$  is bounded, the  $m$ th term tends to infinity near  $\zeta_m$ , and the sum of terms of index greater than  $m$  is bounded if  $a_n \rightarrow 0$  rapidly enough. The technical meaning of "rapidly enough" depends on the distribution of the set  $\{\zeta_n\}$  and the shape of the domain  $H$  near its boundary point 1. Henceforth we shall tacitly assume that the sequence  $\{a_n\}$  meets all requirements of rapid convergence that emerge in the proof.

Because  $G(z) \rightarrow \infty$  as  $z \rightarrow \zeta_n$  in  $\zeta_n H$ , we see from the definition of  $B$  that  $B(z) \rightarrow 1$  as  $z \rightarrow \zeta_n$  in  $\zeta_n H$ .

To establish the existence of the  $H$ -derivative of  $B$  at  $\zeta_n$ , we write

$$g_n(z) = a_n(z + \zeta_n)/(z - \zeta_n)$$

and  $G = g_n + \gamma_n$ . Then

$$B - 1 = \frac{G + 1}{G - 1} - 1 = \frac{2}{G - 1} = \frac{2}{g_n - (1 - \gamma_n)}.$$

Because  $\gamma_n$  is bounded in  $\zeta_n H$ , it follows that

$$\frac{B(z) - 1}{z - \zeta_n} = \frac{2}{a_n(z + \zeta_n) - (z - \zeta_n)(1 - \gamma_n(z))} \rightarrow (a_n \zeta_n)^{-1}$$

as  $z \rightarrow \zeta_n$  through  $\zeta_n H$ .

To see that  $B$  is an inner function, let  $I_n$  denote the arc of length  $1/n^2$  with midpoint  $\zeta_n$  on  $C$ . If  $a_n \rightarrow 0$  rapidly enough, then on each radius of  $D$  whose endpoint lies in at most finitely many of the arcs  $I_n$  and does not belong to the set  $\{\zeta_n\}$ , the series for  $G(z)$  converges uniformly to a function whose value at the endpoint is a finite imaginary number. Therefore the radial limit of  $B$  has modulus 1 almost everywhere on  $C$ .

To show that for an appropriate choice of  $\{a_n\}$  the function  $B$  is a Blaschke product, we denote by  $d_n$  a circular disk tangent from the inside to  $C$  at  $\zeta_n$ , and we suppose that no two of the disks intersect. If  $a_n$  is small enough, then the real part of  $g_n(z)$  lies in the interval  $(-3^{-n}, 0)$  for all  $z$  in  $D \setminus d_n$ . Therefore we may assume that in the set  $\Delta = D - \bigcup d_n$  the real part of  $G(z)$  lies in the interval  $(-1/2, 0)$ . Because  $G(z)$  is bounded away from  $-1$  in  $\Delta$ ,  $B(z)$  is bounded away from 0 in  $\Delta$ . Because every singular function has the radial limit 0 at some point of  $C$ , this implies that  $F$  is a Blaschke product [7].

**REMARK.** For each  $n$  the real part of the radial limit of  $g_n$  is 0 at each point  $\zeta$  in  $C \setminus \{\zeta_n\}$ . By a slight refinement of our argument, the radial cluster set of  $B$  at each point on  $C$  is a subset of  $C$ .

Finally, suppose  $\{\zeta_n\}$  is a countable set of type  $G_\delta$ . Then  $C \setminus \{\zeta_n\}$  is the union of an increasing sequence  $\{F_n\}$  of closed sets; the distance  $\rho_n$  between  $\zeta_n$  and  $F_n$  is positive. The mapping  $w = (z + \zeta_n)/(z - \zeta_n)$  carries the radius vector of a point  $z_0$  on  $C$  onto the circular arc in the left half-plane that is orthogonal to the imaginary axis and has the endpoints  $-1$  and  $(z_0 + \zeta_n)/(z_0 - \zeta_n)$ . On this arc, the maximum modulus of  $w$  is either 1 or  $|z_0 + \zeta_n|/|z_0 - \zeta_n|$ . Therefore, if  $a_n < 2^{-(n+1)}\rho_n$ , then  $|g_n(z)| < 2^{-n}$  on every radius of  $D$  that terminates on  $F_n$ . It follows that at each point of  $C \setminus \{\zeta_n\}$  the function  $G$  has a finite radial limit whose real part is 0. Consequently,  $B$  has a Fatou limit of modulus 1 at each point of  $C$ , and it has the Fatou limit 1 only at the points  $\zeta_n$ .

**3. Preassigned Fatou limits.** The theorem in this section includes the first part of Theorem 1 as a special case. Because for some nontrivial Blaschke products each point of  $C$  is a strong Fatou point, Theorem 2 contains no analogue to the second part of Theorem 1.

**THEOREM 2.** *Let  $\{\zeta_n\}$ ,  $\{\omega_n\}$  and  $H$  be a countable set on  $C$ , a sequence on  $C$ , and a tangent boundary domain at 1. Then there exists a Blaschke product possessing an  $H$ -derivative at each point of  $\{\zeta_n\}$  and having the Fatou limit  $\omega_n$  at  $\zeta_n$  for  $n = 1, 2, \dots$*

Our theorem overlaps with recent results of Cargo [3], who uses the weaker hypothesis  $\omega_n \in D \cup C$ ; but his theorems allow only a finite set of points  $\zeta_n$ .

For each index  $n$  let  $\zeta_n = \exp i\psi_n^*$ . By an inductive process, we shall construct a rapidly decreasing sequence  $\{\delta_n\}$  of positive numbers. For each  $n$  we shall define a short arc  $\Delta_n$  on  $C$  with midpoint  $\zeta_n$ . The elementary building block in our construction will be a two-factor Blaschke product with zeros at the points  $(1 - \delta_n)\exp i(\psi_n \pm \delta_n)$ . Each of the parameters  $\psi_n$  will vary continuously as our construction proceeds, in such a way that  $\exp i\psi_n$  approaches a limit  $\exp i\tilde{\psi}_n$  on  $\Delta_n$ . The following geometric sketch of the construction will put the analytic details in perspective.

The first element of our Blaschke product is the two-zero product  $b_1$ ; for practical purposes its zeros might at the beginning lie at  $(1 - \delta_1) \exp i(\psi_1^* \pm \delta_1)$ ; but to establish as early as possible our induction ritual, we ask the reader to consider first the two-zero product with zeros at  $(1 - \delta) \exp i(\psi_1^* \pm \delta)$ , and to regard  $\delta$  as a parameter in the interval  $[0, \frac{1}{2})$ . As  $\delta \rightarrow 0$ , the product  $b_1$  approaches the value 1 uniformly in the complement of every disk  $|z - \zeta_1| < \epsilon$ . We allow  $\delta$  to begin at 0 and to grow until it reaches some small positive number whose precise value is not relevant to the proof. Thereafter,  $\delta$  remains fixed, in the context of  $b_1$ , and we denote it by  $\delta_1$ . This completes the first half of the first step.

In the second half we replace the constant  $\psi_1^*$  with the variable  $\psi_1$  and allow  $\psi_1$  to increase or decrease until the two-factor Blaschke product with zeros at  $(1 - \delta_1) \exp i(\psi_1 \pm \delta_1)$  takes the value  $\omega_1$  at  $\zeta_1$ . This completes the first step.

Suppose that the  $(n - 1)$ st step has given us  $n - 1$  two-factor products  $b_\nu$  ( $\nu = 1, \dots, n - 1$ ) whose product  $P_{n-1}$  takes the value  $\omega_\nu$  at  $\zeta_\nu$  ( $\nu = 1, \dots, n - 1$ ). We construct  $P_n$  by introducing the two-factor product  $b_n$  whose zeros lie at the points  $(1 - \delta) \exp i(\psi_n^* \pm \delta)$ . As  $\delta \rightarrow 0$  the function  $b_n$  tends to the constant 1 uniformly in the complement of each neighborhood of  $\zeta_n$ . We shall show that as  $\delta$  increases from 0 to  $\delta_n$ , we can adjust the parameters  $\psi_1, \dots, \psi_{n-1}$  continuously so that  $P_n(\zeta_\nu) = \omega_\nu$  for  $\nu = 1, \dots, n - 1$ . Thereafter, as the two zeros of  $b_n$  slide along the circle  $|z| = 1 - \delta_n$ , we can again adjust the parameters  $\psi_1, \dots, \psi_{n-1}$  so as to keep  $P_n(\zeta_\nu)$  constant for  $\nu = 1, \dots, n - 1$ . When  $P_n(\zeta_n)$  reaches the value  $\omega_n$ , the  $n$ th step of the construction is completed.

To ensure that the formal limit of our product satisfies the Blaschke condition, we must choose the positive numbers  $\delta_n$  so that  $\sum \delta_n < \infty$ . In our description we shall focus on a more severe criterion of smallness. Suppose at the end of the  $(n - 1)$ st step of the construction each of the points  $\exp i\psi_\nu$  ( $\nu = 1, \dots, n - 1$ ) lies in the interior of the corresponding arc  $\Delta_\nu$ . We shall show that if  $\delta_n$  is small enough, then at the end of the  $n$ th stage each of the points  $\exp i\psi_\nu$  ( $\nu = 1, \dots, n$ ) still lies in the interior of the corresponding arc  $\Delta_\nu$ .

Suppressing temporarily the index  $n$ , we observe that our two-factor products  $b$  have the form

$$b(\psi, \delta, z) = \frac{(1 - \delta) - e^{-i(\psi + \delta)}z}{1 - (1 - \delta)e^{-i(\psi + \delta)}z} \cdot \frac{(1 - \delta) - e^{-i(\psi - \delta)}z}{1 - (1 - \delta)e^{-i(\psi - \delta)}z}$$

and, consequently, for all  $\theta$ ,

$$b(\psi, \delta, z) = b(\psi + \theta, \delta, ze^{i\theta});$$

in particular,

$$b(\psi_n^* + \theta, \delta_n, z) = b(\psi_n^*, \delta_n, ze^{-i\theta}).$$

From the last equation we deduce that the behavior of  $b(\psi_n, \delta_n, \zeta_n)$  as  $\exp i\psi_n$  traverses the arc  $\Delta_n$  is the same as the behavior of  $b(\psi_n^*, \delta_n, z)$  as  $z$  traverses the arc  $\Delta_n$  in the opposite direction. Taking partial derivatives with respect to  $\theta$  and writing

$$\frac{\partial}{\partial z} b(\psi, \delta, z) = b'(\psi, \delta, z),$$

we obtain the relation

$$\frac{\partial}{\partial \theta} b(\psi_n^* + \theta, \delta_n, z) = ie^{-i\theta} z b'(\psi_n^*, \delta_n, ze^{-i\theta}).$$

As  $z$  traverses the circle  $C$ , the value  $b(\psi, \delta, z)$  traverses  $C$  twice; moreover, most of the movement of  $b(\psi, \delta, z)$  takes place while  $z$  traverses a short arc of  $C$  with midpoint  $\exp i\psi$ . Invoking the convention that  $\arg b(\psi, \delta, e^{i\psi}) = 0$ , we define  $\Delta(\psi, \delta)$  as the subarc of  $C$  along which  $\arg b(\psi, \delta, z)$  increases from  $-3\pi/2$  to  $3\pi/2$ . Let  $2u(\delta)$  denote the length of  $\Delta(\psi, \delta)$ . It is easy to verify that there exist positive constants  $K_1$  and  $K_2$  such that  $u(\delta) < K_1\delta$  and

$$(3.1) \quad |b'(\psi, \delta, z)| > K_2/\delta$$

whenever  $z = \exp i(\psi + \theta)$  and  $|\theta| < 2u(\delta)$ .

Once  $\delta_n$  is chosen, we define the arc  $\Delta_n$  by the rule

$$\Delta_n = \Delta(\psi_n^*, \delta_n) = \{e^{i\psi} : |\psi - \psi_n^*| < u(\delta_n)\}.$$

From our definition of  $\Delta_1$  it follows that at the end of the first step we can choose the point  $\exp i\psi_1$  so that it lies in  $\Delta_1$ .

Suppose now that at the end of the  $(n-1)$ st step the points  $\exp i\psi_\nu$  ( $\nu = 1, \dots, n-1$ ) lie in the interiors of the corresponding arcs  $\Delta_\nu$  and  $P_{n-1}(\zeta_k) = \omega_k$  for  $k = 1, \dots, n-1$ . During the first half of the  $n$ th step, the variable  $\psi$  in our new factor  $b_n$  has the fixed value  $\psi_n^*$  and the variable  $\delta$  increases from 0 to some  $\delta_n$ . Meanwhile, the behavior of the variables  $\psi_\nu$  ( $\nu = 1, \dots, n-1$ ) is governed by the  $n-1$  equations  $P_n(\zeta_k) = \omega_k$ . If we write these equations in detail, they take the form

$$\left[ \prod_{\nu=1}^{n-1} b(\psi_\nu, \delta_\nu, \zeta_k) \right] b(\psi_n^*, \delta, \zeta_n) = \omega_k.$$

We regard the parameters  $\psi_\nu$  as dependent on  $\delta$ , take the logarithmic partial derivative with respect to  $\delta$ , and obtain the equations

$$(3.2) \quad \sum_{\nu=1}^{n-1} \frac{\partial b(\psi_\nu, \delta_\nu, \zeta_k)/\partial \psi_\nu}{b(\psi_\nu, \delta_\nu, \zeta_k)} \cdot \frac{\partial \psi_\nu}{\partial \delta} + \frac{\partial b(\psi_n^*, \delta, \zeta_k)/\partial \delta}{b(\psi_n^*, \delta, \zeta_k)} = 0.$$

The last term on the left has a bound that depends only on  $\psi_n^* - \psi_k^*$ . In the coefficient of  $\partial \psi_\nu / \partial \delta$ , the denominator has modulus 1; as long as  $\exp i\psi_\nu$  lies on  $\Delta_\nu$ , it follows from (3.1) that the modulus of the numerator is greater than  $K_2/\delta_k$  if  $\nu = k$ ; if  $\nu \neq k$ , the numerator has a bound independent of  $\delta$ . It follows that if we have chosen  $\delta_{n-1}$  small enough, then the Jacobian of the system (3.2) is bounded away from 0 as long as  $\delta$  is small enough and  $\exp i\psi_\nu$  lies on  $\Delta_\nu$  for  $\nu = 1, \dots, n-1$ . This implies that each of the partial derivatives  $\partial \psi_\nu / \partial \delta$  is a bounded function of  $\delta$ . We deduce that at the end of the first half of the  $n$ th step the points  $\exp i\psi_\nu$  ( $\nu = 1, \dots, n-1$ ) lie as near as we like to their positions at the beginning of the  $n$ th step, provided we have chosen  $\delta_n$  small enough.

For the second half of the  $n$ th step we must replace the partial derivatives  $\partial \psi_\nu / \partial \delta$  in (3.2) with  $\partial \psi_\nu / \partial \psi_n$  and the last term with

$$\frac{\partial b(\psi_n, \delta_n, \zeta_k)/\partial \psi_n}{b(\psi_n, \delta_n, \zeta_k)}.$$

Again, the last term is a bounded function. The second half of the  $n$ th stage ends when  $P_n(\xi_n)$  reaches the value  $\omega_n$ . Because this requires an increase or decrease in  $\psi_n$  less than  $u(\delta_n)$ , we see that if we have chosen  $\delta_n$  small enough, then at the end of the  $n$ th stage the point  $\exp i\psi_n$  lies in  $\Delta_\nu$  for  $\nu = 1, \dots, n$ .

It is now obvious that if  $\delta_n \rightarrow 0$  rapidly enough, then, as the construction continues, each of our parameters  $\psi_n$  approaches a limit  $\tilde{\psi}_n$ ; the formal product

$$B(z) = \prod_1^\infty b(\tilde{\psi}_n, \delta_n, z)$$

is a Blaschke product, and  $B(\xi_k) = \omega_k$  for  $k = 1, 2, \dots$ .

Finally, let

$$B_n(z) = \prod_{\nu=1}^n b(\tilde{\psi}_\nu, \delta_\nu, z).$$

Then for each index  $k$  the sequence  $\{[B_n(z) - B_n(\xi_k)]/[z - \xi_k]\}$  of difference quotients converges for each  $z$  in  $D$ , and the convergence is uniform in the domain  $\xi_k H$  if  $\delta_n \rightarrow 0$  rapidly enough. This concludes the proof of Theorem 2.

**4. Uncountable sets of strong Fatou-1-points.** Because an inner function has a finite angular derivative at no more than countably many Fatou-1-points, we now abandon the angular derivative and its generalizations.

**THEOREM 3.** *There exists a Blaschke product with uncountably many strong Fatou-1-points.*

Let  $M$  denote the image of the classical Cantor set under the obvious mapping of the interval  $[0, 1]$  onto the arc of  $C$  whose length is 1 and whose midpoint is 1, and let  $d\mu$  denote the standard singular distribution of unit mass on  $M$ . We take the liberty of writing  $d\mu(t)$  instead of  $d\mu(e^{it})$ , and we define the function

$$G(z) = u(z) + iv(z) = \int_{-\pi}^{\pi} \frac{z + e^{it}}{-z - e^{it}} d\mu(t).$$

We shall show that each point of  $M$  is a strong Fatou-1-point of the function  $B = (G + 1)/(G - 1)$ . Because  $G$  has a holomorphic extension across the set  $C \setminus M$  and its real part is 0 everywhere on this set, it will follow that  $B$  is a Blaschke product.

Because holomorphic mappings of  $D$  into  $D$  never increase non-Euclidean distances, it is sufficient to prove that  $B$  maps each radius terminating on  $M$  into a Stolz path terminating at 1. This is equivalent to proving that on each radius terminating on  $M$  the value of  $G(z)$  tends to infinity in a wedge whose sides lie in the left half-plane.

We regard  $M$  as the intersection of closed sets  $M_n$ , each consisting of  $2^n$  arcs of length  $3^{-n}$ . Each of these arcs supports a mass  $2^{-n}$  of the distribution  $d\mu$ .

Let  $\xi = e^{it_0}$  denote a point in  $M$  and write  $z = r\xi$  ( $1/2 \leq r < 1$ ). Then

$$u(z) = \int_{-\pi}^{\pi} \frac{(r^2 - 1) d\mu(t)}{1 - 2r \cos(t - t_0) + r^2} \quad \text{and} \quad v(z) = \int_{-\pi}^{\pi} \frac{-2r \sin(t - t_0) d\mu(t)}{1 - 2r \cos(t - t_0) + r^2}.$$

We recall the identity

$$(4.1) \quad 1 - 2r \cos(t - t_0) + r^2 = (1 - r)^2 + 4r \sin^2(t - t_0)/2.$$

Let  $n = n(r)$  denote the integer for which  $3^{-n} < 1 - r \leq 3^{1-n}$ , and let  $I(z)$  denote the component of  $M_n$  that contains the point  $\zeta$ . Then  $|t - t_0| < 1 - r$  on  $I(z)$ , so the right side of (4.1) is less than  $(1 + r)(1 - r)^2$ . Sacrificing all contributions to  $u(z)$  except that from  $I(z)$ , we obtain the inequality

$$|u(z)| > 1/2^n(1 - r) > 3^{-1}(3/2)^n.$$

To obtain a comparable upper bound on  $|v(z)|$ , we consider first the contribution from  $I(z)$ . Here

$$2r|\sin(t - t_0)| < 2r(1 - r) < (1 + r)(1 - r),$$

and therefore the contribution to  $v(z)$  has absolute value less than  $|u(z)|$ .

To deal with the remainder of  $M_n$ , we consider for each of the indices  $k = 1, 2, \dots, n$  the  $2^k - 1$  components of  $M_k$  that do not contain  $\zeta$ . One of these  $2^k - 1$  components is nearer to  $\zeta$  than all the others; we denote it by  $I_k$ . Clearly,

$$M \subset M_n \subset I(z) \cup I_n \cup I_{n-1} \cup \dots \cup I_1.$$

The angular distance between  $\zeta$  and  $I_k$  is at least  $3^{-k}$ . Therefore the estimate

$$\frac{2r|\sin(t - t_0)|}{4r \sin^2(t - t_0)/2} = \left| \cot \frac{t - t_0}{2} \right| < \frac{2}{|t - t_0|} < \frac{2}{3^{-k}}$$

is valid everywhere on  $I_k$ , and because the mass of  $d\mu$  on  $I_k$  is  $2^{-k}$ , the contribution to  $v(z)$  from  $I_k$  has absolute value less than  $2^{1-k}3^k = 2(3/2)^k$ . The sum over  $k$  of these quantities is

$$2(3/2 + \dots + (3/2)^n) < 6(3/2)^n < 18|u(z)|.$$

It follows that  $|v(z)| < 19|u(z)|$ , and Theorem 3 is proved.

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